Hopf algebras of canonical commutation relations

G. Sardanashvily

Department of Theoretical Physics, Physics Faculty, Moscow State University, 117234 Moscow, Russia
E-mail: sard@grav.phys.msu.su
URL: http://webcenter.ru/~sardan/

Abstract
Given a Heisenberg algebra A of canonical commutation relations modelled over an infinite-dimensional nuclear space, a Hopf algebra of its quantum deformations is also an algebra of canonical commutation relations whose Fock representation recovers some non-Fock representation of A.

1 Introduction

By virtue of the well-known Stone–von Neumann uniqueness theorem, all irreducible representations of the canonical commutation relations (henceforth the CCR) of finite degree of freedom are equivalent. On the contrary, the infinite-dimensional CCR possess many non-equivalent irreducible representations (see [1] for a survey). Here, we restrict our consideration to the CCR modelled over a nuclear space. They include the CCR of finite degrees of freedom, but we focus on the infinite-dimensional CCR. In particular, this is the case of field theory [5].

Let A be the Heisenberg algebra of the CCR modelled over a nuclear space. Since A is a Lie algebra, one can associate to A a Hopf algebra, regarded as an algebra of q-deformed CCR (see [3] for the case of finite-dimensional CCR). We show that this Hopf algebra is the enveloping algebra of another CCR algebra $A_{q,c}$. Moreover, A and $A_{q,c}$ possess the same set of representations. Herewith, operators of the Fock representation of $A_{q,c}$ carry out some non-Fock representation of A.

2 The nuclear CCR

Let us recall the notion of a nuclear space (see, e.g., [4]). Let a complex vector space V be provided with a countable set of non-degenerate Hermitian forms $\langle.|.\rangle_k$, $= 1,\ldots$, such that

$$\langle v|v\rangle_1 \leq \cdots \leq \langle v|v\rangle_k \leq \cdots$$

for all $v \in V$. Let V be complete in the topology defined by the set of norms $\|.\|_k^{1/2} = \langle.|.\rangle_k$. Then V is called a countably Hilbert space. Let V_k denote the completion of V with respect to the norm $\|.\|_k$. There is the chain of injections $V_1 \supset V_2 \supset \cdots V_k \supset \cdots$, and $V = \bigcap_k V_k$. Let T^n_m, $m \leq n$, be a prolongation of the map $V_n \supset V \ni v \mapsto v \in V \subset V_m$ to the continuous map of V_n onto a dense subset of V_m. A countably Hilbert space V is called a nuclear space if, for any m, there exists n such that T^m_n is a nuclear map, i.e.,

$$T^n_m(v) = \sum_i \lambda_i \langle v|v^i_n\rangle_{V_n} v^i_m,$$

where: (i) $\{v^i_n\}$ and $\{v^i_m\}$ are bases for the Hilbert spaces V_n and V_m, respectively, (ii) $\lambda_i \geq 0$, and (iii) the series $\sum \lambda_i$ converges. Note that a Hilbert space is not nuclear, unless it is finite-dimensional.

Let V be a real nuclear space provided with still another non-degenerate Hermitian form $\langle .|.\rangle$, which is separately continuous. This form makes V to a separable pre-Hilbert space. Let us consider the group G of the triples (v_1, v_2, λ) of elements v_1, v_2 of V and complex numbers λ of unit modulus which are subject to multiplications

$$(v_1, v_2, \lambda)(v'_1, v'_2, \lambda') = (v_1 + v'_1, v_2 + v'_2, \exp[i\langle v_2, v'_1\rangle]\lambda\lambda'). \tag{1}$$

It is a Lie group whose group space is a nuclear manifold modelled over

$$W = V \oplus V \oplus \mathbb{R}. \tag{2}$$

Let us denote $T(v) = (v, 0, 0)$ and $P(v) = (0, v, 0)$. Then the multiplication law (1) takes the form

$$\begin{aligned} &T(v)T(v') = T(v + v'), \qquad P(v)P(v') = P(v + v'), \\ &P(v)T(v') = \exp[i\langle v|v'\rangle]T(v')P(v). \end{aligned} \tag{3}$$

Written in this form, G is called the Weyl CCR group.

The Lie algebra of the nuclear Lie group G is the above mentioned Heisenberg algebra A. It is generated by the Hermitian elements I, $\phi(v)$, $\pi(v)$, $v \in V$, which obey the commutation relations

$$[\phi(v), I] = [\pi(v), I] = [\phi(v), \phi(v')] = [\pi(v), \pi(v')] = 0, \tag{4}$$

$$[\pi(v), \phi(v')] = -i\langle v|v'\rangle I. \tag{5}$$

Given a countable orthonormal basis $\{v_i\}$ for the pre-Hilbert space V, the CCR (4) – (5) take the form

$$[\phi(v_j), \phi(v_k)] = [\pi(v_k), \pi(v_j)] = 0, \qquad [\pi(v_j), \phi(v_k)] = -i\delta_{jk} I.$$

One also introduces the creation and annihilation operators

$$a^\pm(v) = \frac{1}{\sqrt{2}}[\phi(v) \mp i\pi(v)]. \tag{6}$$

They obey the conjugation rule $(a^\pm(v))^* = a^\mp(v)$ and the commutation relations

$$[a^-(v), a^+(v')] = \langle v|v'\rangle I, \qquad [a^+(v), a^+(v')] = [a^-(v), a^-(v')] = 0.$$

3 Hopf algebras of the CCR

Let us consider the tensor algebra $\otimes W$ of the vector space W (2) generated by elements $\phi(v)$, $\pi(v)$ and I. It is provided with a unique Hopf algebra structure, characterized by the comultiplication

$$\Delta(w) = w \otimes \mathbf{1} + \mathbf{1} \otimes w, \qquad w \in W,$$

the counit $\epsilon(w) = 0$, the antipode $S(w) = -w$, and the universal matrix $R = \mathbf{1} \otimes \mathbf{1}$. It is a cocommutative quasi-triangular Hopf algebra, called the classical Hopf algebra.

Let $\overline{A}$ be the enveloping algebra of the Heisenberg CCR algebra A. It is the quotient of the tensor algebra $\otimes W$ by the commutation relations (4) – (5), written with respect to the tensor product $\otimes$, and by the relation

$$I \otimes I = I. \tag{7}$$

The $\overline{A}$ inherits the structure of the classical Hopf algebra on $\otimes W$. We denote it $B_{\rm cl}(A)$.

Now let us consider the quotient $\overline{A}_{q,c}$ of the tensor algebra $\otimes W$ by the relations (4), (7) and the commutation relations

$$[\pi(v), \phi(v')] = -i\langle v|v'\rangle \frac{q^{cI} - q^{-cI}}{c(q - q^{-1})}, \tag{8}$$

where q and c are strictly positive real numbers. Due to the relation (7), the right-hand side of the relations (8) is well defined on $\otimes W$, and we have

$$[\pi(v), \phi(v')] = -i\langle v|v'\rangle \frac{q^{c} - q^{-c}}{c(q - q^{-1})} I. \tag{9}$$

Hence, $\overline{A}_{q,c}$ is the enveloping algebra of the Heisenberg CCR algebra $A_{q,c}$ given by the commutation relations (4) and (9). This CCR algebra is modelled over the same nuclear space V, but provided with the Hermitian form

$$\langle v|v'\rangle_{q,c} = C_{q,c}\langle v|v'\rangle, \qquad C_{q,c} = \frac{q^{c} - q^{-c}}{c(q - q^{-1})}. \tag{10}$$

The enveloping algebra $\overline{A}_{q,c}$ admits both the structure of the classical Hopf algebra $B_{\rm cl}(A_{q,c})$ and the Hopf algebra $B(A_{q,c})$, which differs from the classical one in the comultiplication law

$$\Delta(\phi(v)) = \phi(v) \otimes q^{cI/2} + q^{-cI/2} \otimes \phi(v), \qquad \Delta(\pi(v)) = \pi(v) \otimes q^{cI/2} + q^{-cI/2} \otimes \pi(v),$$
$$\Delta(I) = I \otimes \mathbf{1} + \mathbf{1} \otimes I.$$

One can think of $B(A_{q,c})$ as being a Hopf algebra of the q-deformed CCR. It is readily observe that, if $c = 1$, the CCR algebras A and $A_{q,1}$ coincide for any q, but the Hopf algebra $B(A_{q,1})$ differs from the classical one $B_{\rm cl}(A_{q,1}) = B_{\rm cl}(A)$. If $q = 1$, then $A_{1,c} = A$ and $B(A_{1,c}) = B_{\rm cl}(A)$ for any c.

Since the Hopf algebra $B(A_{q,c})$ is the enveloping algebra of the CCR algebra $A_{q,c}$, its representations are determined in full by representations of $A_{q,c}$ Let us compare the representations of the CCR algebras A and $A_{q,c}$.

4 Representations of the nuclear CCR

The CCR group G contains two Abelian subgroups T and P. Following the representation algorithm in [2], we first construct representations of the nuclear Abelian group T [5].

Its cyclic strongly continuous unitary representation ρ in a Hilbert space $(E, \langle .|.\rangle_E)$ with a (normed) cyclic vector $\theta \in E$ defines the complex function

$$Z(v) = \langle \rho(T(v))\theta|\theta\rangle_E$$

on V. This function is continuous and positive-definite, i.e., $Z(0) = 1$ and

$$\sum_{i,j} Z(v_i - v_j)\overline{c}_i c_j \geq 0$$

for any finite set $v_1, \ldots, v_m$ of elements of V and arbitrary complex numbers $c_1, \ldots, c_m$. By virtue of the well-known Bochner theorem, such a function on a nuclear space V is the Fourier transform

$$Z(v) = \int \exp[i\langle v, u\rangle]\mu \tag{11}$$

of a positive measure μ of total mass 1 on the topological dual V' of V. Then the above mentioned representation ρ of T can be given by the operators

$$T_Z(v)f(u) = \exp[i\langle v, u\rangle]f(u) \tag{12}$$

in the Hilbert space $L^2(V', \mu)$ of classes of μ-equivalent square integrable complex functions $f(u)$ on V'. The cyclic vector θ of this representation is the μ-equivalence class $\theta \approx_\mu 1$ of the constant function $f(u) = 1$. Conversely, every positive measure μ of total mass 1 on the dual V' of V (and, consequently, every continuous positive-definite function $Z(v)$ on V) defines a cyclic strongly continuous unitary representation (12) of the nuclear group T. We agree to call Z a generating function of this representation. One can show that distinct generating functions Z and Z' determine equivalent representations T_Z and $T_{Z'}$ (12) of T in the Hilbert spaces $L^2(V', \mu)$ and $L^2(V', \mu')$ iff they are the Fourier transform of equivalent measures on V'.

The representation T_Z (12) of the group T can be extended to the CCR group G if the measure μ possesses the following property. Let u_v, $v \in V$, denote an element of V' given by the condition

$$\langle v', u_v\rangle = \langle v'|v\rangle, \qquad \forall v' \in V. \tag{13}$$

These elements form the image of the monomorphism $V \to V'$ determined by the Hermitian form $\langle .|.\rangle$ on V. Let the measure μ in (11) remain equivalent under translations $u \mapsto u + u_v$ of V' by any element u_v of $V \subset V'$, i.e.,

$$\mu(u + u_v) = a^2(v, u)\mu(u), \qquad \forall u_v \in V \subset V', \tag{14}$$

where a function $a(v, u)$ is square μ-integrable and strictly positive almost everywhere on V'. This function fulfils the relations

$$a(0, u) = 1, \qquad a(v + v', u) = a(v, u)a(v', u + u_v). \tag{15}$$

A measure on V' obeying the condition (14) is called translationally quasi-invariant. Let the generating function Z of a cyclic strongly continuous unitary representation of the nuclear group T be the Fourier transform (11) of such a measure μ on V'. Then the representation (12) of T is extended to the representation of the nuclear CCR group G in the Hilbert space $L^2(V', \mu)$ by operators

$$P_Z(v)f(u) = a(v, u)f(u + u_v). \tag{16}$$

Moreover, one can show that if μ' is a μ-equivalent positive measure of total mass 1 on V', it is also translationally quasi-invariant and provides an equaivalent representation of G.

A strongly continuous unitary representation T_Z (12), P_Z (16) of the nuclear CCR group G implies a representation of its Lie algebra A by (unbounded) operators

$$I = \mathbf{1}, \quad \phi(v)f(u) = \langle v, u\rangle f(u), \quad \pi(v)f(u) = -i(\delta_v + \eta(v, u))f(u), \tag{17}$$

$$\delta_v f(u) = \lim_{\alpha\to 0} \alpha^{-1}[f(u + \alpha u_v) - f(u)], \qquad \alpha \in \mathbb{R},$$

$$\eta(v, u) = \lim_{\alpha\to 0} \alpha^{-1}[a(\alpha v, u) - 1], \tag{18}$$

in the same Hilbert space $L^2(V', \mu)$. With the aid of the formulas

$$\begin{aligned}
&\delta_v\delta_{v'} = \delta_{v'}\delta_v, \qquad \delta_v(\eta(v', u)) = \delta_{v'}(\eta(v, u)),\\
&\delta_v = -\delta_{-v}, \qquad \delta_v(\langle v', u\rangle) = \langle v'|v\rangle,\\
&\eta(0, u) = 0, \quad \forall u \in V', \qquad \delta_v\theta = 0, \quad \forall v \in V,
\end{aligned}$$

derived from the relations (15), it is easily justified that the operators (17) fulfil the Heisenberg CCR (4).

Gaussian measures exemplify a physically relevant class of translationally quasi-invariant measures on the dual V' of a nuclear space V. The Fourier transform of a Gaussian measure reads

$$Z(v) = \exp\left[-\frac{1}{2}M(v)\right], \tag{19}$$

where $M(v)$ is a seminorm on V' called the covariance form. Let μ_K denote a Gaussian measure on V' whose Fourier transform is the generating function

$$Z_K = \exp[-\frac{1}{2}M_K(v)] \tag{20}$$

with the covariance form $M_K(v) = \langle K^{-1}v|K^{-1}v\rangle$, where K is a bounded invertible operator in the Hilbert completion $\widetilde{V}$ of V with respect to the Hermitian form $\langle .|.\rangle$. The Gaussian measure μ_K is translationally quasi-invariant:

$$\begin{aligned} &\mu_K(u+u_v) = a_K^2(v,u)\mu_K(u), \\ &a_K(v,u) = \exp[-\frac{1}{4}M_K(Cv) - \frac{1}{2}\langle Cq, u\rangle], \end{aligned} \tag{21}$$

where $C = KK^*$ is a bounded Hermitian operator in $\widetilde{V}$.

Let us construct the representation of the CCR algebra A determined by the generating function Z_K (20). Substituting the function (21) into the formula (18), we find

$$\eta(v,u) = -\frac{1}{2}\langle Cv, u\rangle.$$

Hence, the operators $\phi(v)$ and $\pi(v)$ (17) take the form

$$\phi(v) = \langle v, u\rangle, \qquad \pi(v) = -i(\delta_v - \frac{1}{2}\langle Cv, u\rangle). \tag{22}$$

Accordingly, the creation and annihilation operators (6) read

$$a^{\pm}(v) = \frac{1}{\sqrt{2}}[\mp\delta_v \pm \frac{1}{2}\langle Cv, u\rangle + \langle v, u\rangle]. \tag{23}$$

In particular, let us put $K = \sqrt{2}\cdot\mathbf{1}$. Then the generating function (20) takes the form

$$Z_{\rm F}(v) = \exp[-\frac{1}{4}\langle v|v\rangle], \tag{24}$$

and determines the Fock representation of the CCR algebra A by the operators

$$\begin{aligned} &\phi(v) = \langle v, u\rangle, \qquad \pi(v) = -i(\delta_v - \langle v, u\rangle), \\ &a^+(v) = \frac{1}{\sqrt{2}}[-\delta_v + 2\langle v, u\rangle], \qquad a^-(v) = \frac{1}{\sqrt{2}}\delta_v. \end{aligned}$$

Note that the Fock representation up to an equivalence is characterized by the existence of a cyclic vector θ such that

$$a^-(v)\theta = 0, \qquad \forall v \in V. \tag{25}$$

An equivalent condition is that there exists the particle number operator N possessing a lower bounded spectrum. This operator is defined by the conditions

$$[N, a^\pm(v)] = \pm a^\pm(v)$$

up to a summand $\lambda\mathbf{1}$. With respect to a countable orthonormal basis $\{v_k\}$, it is given by the sum

$$N = \sum_k a^+(v_k)a^-(v_k).$$

A glance at the expression (23) shows that the condition (25) does not hold, unless Z_K is $Z_{\rm F}$ (24). For instance, the particle number operator in the representation (23) reads

$$N = \sum_j a^+(v_j)a^-(v_j) = \sum_j [-\delta_{v_j}\delta_{v_j} + C^j_k\langle v_k, u\rangle\partial_{v_j} +$$
$$(\delta_{km} - \frac{1}{4}C^j_k C^j_m)\langle v_k, u\rangle\langle v_m, u\rangle - (\delta_{jj} - \frac{1}{2}C^j_j)].$$

One can show that this operator is defined and is lower bounded only if the operator C is a sum of the scalar operator $2\cdot\mathbf{1}$ and a nuclear operator in $\widetilde V$. For instance, the generating function

$$Z_c(v) = \exp[-\frac{c^2}{2}\langle v|v\rangle], \qquad c^2 \neq \frac{1}{2},$$

determines a non-Fock representation of the nuclear CCR.

At the same time, the non-Fock representation (22) of the CCR algebra (4) is the Fock representation

$$\phi_K(v) = \phi(v) = \langle v, u\rangle,$$
$$\pi_K(v) = \pi(S^{-1}v) = -i(\delta^K_v - \frac{1}{2}\langle v, u\rangle), \qquad \delta^K_v = \delta_{S^{-1}v},$$

of the CCR algebra $\{\phi_K(v), \pi_K(v), I\}$, where

$$[\phi_K(v), \pi_K(v)] = i\langle K^{-1}v|K^{-1}v'\rangle I.$$

Bearing in mind this fact, turn now to the CCR algebra $A_{q,c}$ in Section 3. Comparing the commutation relations (5) and (9), one can show that, given a representation ρ of the CCR algebra A, the CCR algebra $A_{q,c}$ admits a representation $\rho_{q,c}$ by the operators

$$\rho_{q,c}(\phi(v)) = \rho(\phi(v)), \qquad \rho_{q,c}(\pi(v)) = \rho(\pi(C_{q,c}v)), \qquad \rho_{q,c}(I) = \rho(I) = \mathbf{1},$$

where $C_{q,c}$ is the real number given by the expression (10). For instance, if ρ is the Fock representation of the CCR algebra A, the representation $\rho_{q,c}$ is not equivalent to the Fock representation of the CCR algebra $A_{q,c}$, unless V is finite-dimensional.

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